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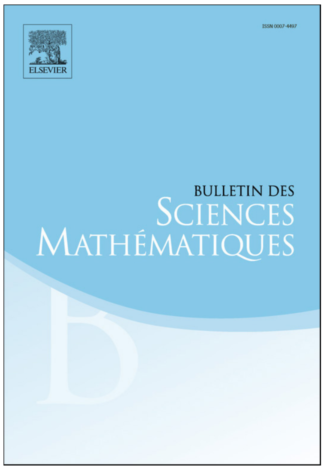
Sharp seasonal threshold property for cooperative population dynamics with concave nonlinearities



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Sharp seasonal threshold property for cooperative population dynamics with concave nonlinearities

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Abstract

We consider a biological population whose environment varies periodically in time, exhibiting two very different "seasons": one is favorable and the other one is unfavorable. For monotone differential models with concave nonlinearities, we address the following question: the system's period being fixed, under what conditions does there exist a critical duration for the unfavorable season? By "critical duration" we mean that above some threshold, the population cannot sustain and extincts, while below this threshold, the system converges to a unique periodic and positive solution. We term this a "sharp seasonal threshold property" (SSTP, for short).

Building upon a previous result, we obtain sufficient conditions for SSTP in any dimension and apply our criterion to a two-dimensional model featuring juvenile and adult populations of insects.

Keywords: dynamical systems; periodic forcing; seasonality; population dynamics;
2010 Mathematics Subject Classification: 15B48; 34D23; 34C25; 37C65; 92D25;

1 Introduction

We study differential dynamical systems arising from nonlinear periodic positive differential equations of the form

$$\frac{dx}{dt} = F(t,x), \tag{1.1}$$

where F is monotone and concave (in x). These systems exhibit well-known contraction properties when F is continuous (see [7], [9], [10]). We extend in Theorem 1 these properties to non-linearities that are only piecewise-continuous in time. This extension is motivated by the study of typical seasonal systems in population dynamics.

We denote by $\theta \in [0,1]$ the proportion of the year spent in unfavorable season. Then, we convene that time t belongs to an unfavorable (resp. a favorable) season if $nT \leq t < (n+\theta)T$ (resp. if $(n+\theta)T \leq t < (n+1)T$) for some $n \in \mathbb{Z}_+$. In other words, we study the solutions to:

$$\frac{dX}{dt} = G(\pi_\theta(t), X), \quad \pi_\theta(t) = \begin{cases} \pi^U \text{ if } \frac{t}{T} - \lfloor \frac{t}{T} \rfloor \in [0,\theta), \\ \pi^F \text{ if } \frac{t}{T} - \lfloor \frac{t}{T} \rfloor \in [\theta,1), \end{cases} \tag{1.2}$$

for some $G : \mathcal{P} \times \mathbb{R}^N \to \mathbb{R}^N$, with $\pi^U, \pi^F \in \mathcal{P}$ where $\mathcal{P}$ is the parameter space. We are looking for conditions ensuring that a sharp seasonal threshold property holds, that is:

$$\exists \theta_* \in [0,1] \text{ such that } \begin{cases} \text{if } \theta < \theta_*, \exists! q : \mathbb{R}_+ \to \mathbb{R}^N, T\text{-periodic}, q \gg 0 \text{ and} \\ \forall X_0 \in \mathbb{R}_+^N \backslash \{0\}, X \text{ converges to } q, \\ \text{if } \theta > \theta_*, \forall X_0 \in \mathbb{R}_+^N,\ X \text{ converges to } 0. \end{cases} \tag{SSTP}$$

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THE STABILITY ANALYSIS OF BRAIN LACTATE KINETICS

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Abstract. Our aim in this article is to study properties of a generalized dynamical system modeling brain lactate kinetics, with N neuron compartments and A astrocyte compartments. In particular, we prove the uniqueness of the stationary point and its asymptotic stability. Furthermore, we check that the system is positive and cooperative.

1. Introduction.

The system of ODE's

$$\begin{aligned} \frac{dx}{dt} &= J - T\left(\frac{x}{k+x} - \frac{y}{k'+y}\right), \quad T,\ k,\ k',\ J > 0, \\ \epsilon\frac{dy}{dt} &= F(L-y) - T\left(\frac{y}{k'+y} - \frac{x}{k+x}\right), \quad \epsilon,\ F,\ L > 0, \end{aligned} \tag{1}$$

where ϵ is a small parameter, was proposed and studied as a model for brain lactate kinetics (see [11, 8, 9, 10]). In this context, $x = x(t)$ and $y = y(t)$ correspond to the lactate concentrations in an interstitial (i.e., extra-cellular) domain and in a capillary domain, respectively. Furthermore, the nonlinear term $T(\frac{x}{k+x} - \frac{y}{k'+y})$ stands for a co-transport through the brain-blood boundary (see [7]). Finally, J and F are forcing and input terms, respectively, assumed frozen. The model has a unique stationary point which is asymptotically stable. Recently, in [2, 12], a PDE's system obtained by adding diffusion of lactate was introduced. The authors proved existence and uniqueness of nonnegative solutions and obtained linear stability results. A more general ODE's model for brain lactate kinetics, where the intracellular compartment splits into neuron and astrocyte, was considered in [8, 9]. It displays

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Key words and phrases. Brain lactate kinetics, cooperative dynamical systems, asymptotic stability.

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GLOBAL DYNAMICS OF A PIECEWISE SMOOTH SYSTEM FOR BRAIN LACTATE METABOLISM

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Abstract. In this article, we study a piecewise smooth dynamical system inspired by a previous reduced system modeling compartimentalized brain metabolism. The piecewise system allows the introduction of an autoregulation induced by a feedback of the extra-cellular or capillary Lactate concentrations on the Capillary Blood Flow. New dynamical phenomena are uncovered and we discuss existence and nature of two equilibrium points, attractive segment, boundary equilibrium and periodic orbits depending of the Capillary Blood Flow.

1. Introduction

The nonlinear system of ODEs defined as follows:

$$
\begin{aligned}
\frac{dx}{dt} &= J - T\left(\frac{x}{k+x} - \frac{y}{k'+y}\right) \qquad T,\ k,\ k',\ J > 0,\\
\frac{dy}{dt} &= F(L-y) + T\left(\frac{x}{k+x} - \frac{y}{k'+y}\right) \qquad F,\ L > 0,
\end{aligned}
\tag{1.1}
$$

where $(x, y) \in \mathbb{R}^2_+$ was first proposed and studied as a model for coupled energy metabolism between Neuron-Astrocyte and Capillary by [Costalat, Françoise, Guillevin, Lahutte-Auboin] (see [4, 10, 11, 12]). In this context, $x = x(t)$ and $y = y(t)$ correspond to the Lactate concentrations in an interstitial (i.e. extra-cellular) domain and in a Capillary domain, respectively. Furthermore, the nonlinear term $T(\frac{x}{k+x} - \frac{y}{k'+y})$ stands for a co-transport through the Brain-Blood Boundary (see [9]). The forcing term J represents the Lactate flux in the intracellular domain. Furthermore the input F stands for the Capillary Blood Flow through capillaries from arterial to venous, and L represents arterial Lactate. In these previous articles, different time scales were considered on the evolution of the two variables and the asymptotics of fast-slow dynamical systems was used (see also a more recent reference [6]). Here, our results are independent of this scaling. Recently, in [13, 7], a PDE's system obtained by adding diffusion of Lactate was introduced. The authors proved existence and uniqueness of nonnegative solutions and obtained linear stability results. In system (1.1) the forcing term J and input terms F are assumed frozen.

In [12], the physiological domain was discussed in terms of bounds on the Lactate concentrations x and y. It is natural to push further this study with the introduction of a kind of autoregulation of the system induced by a feedback (for instance of Astrocytes on the Capillary) of the two concentrations (x or y) on the Capillary Blood Flow F. This is discussed in this article where the autoregulation is represented by a piecewise variation of

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Key words and phrases. Piecewise Smooth System; Qualitative Analysis; Modeling.

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