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Comparison Geometry for Integral Bakry–Émery Ricci Tensor Bounds

Jia-Yong Wu[1]

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Abstract We prove mean curvature and volume comparison estimates on smooth metric measure spaces when their integral Bakry–Émery Ricci tensor bounds, extending Wei–Wylie's comparison results to the integral case. We also apply comparison results to get diameter estimates, eigenvalue estimates, and volume growth estimates on smooth metric measure spaces with their normalized integral smallness for Bakry–Émery Ricci tensor. These give generalizations of some work of Petersen–Wei, Aubry, Petersen–Sprouse, Yau and more.

Keywords Bakry–Émery Ricci tensor · Smooth metric measure space · Integral curvature · Comparison theorem · Diameter estimate · Eigenvalue estimate · Volume growth estimate

Mathematics Subject Classification Primary 53C20

1 Introduction and Main Results

In [18], Petersen and Wei generalized the classical relative Bishop–Gromov volume comparison to a situation where one has an integral bound for the Ricci tensor. Let's briefly recall their results. Given an n-dimensional complete Riemannian manifold M, for each $x \in M$ let $\lambda(x)$ be the smallest eigenvalue for the Ricci tensor $\mathrm{Ric} : T_x M \to T_x M$, and

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Calculus of Variations

Gradient shrinking Ricci solitons of half harmonic Weyl curvature

Jia-Yong Wu[1] · Peng Wu[2] · William Wylie[3]

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Abstract

Gradient Ricci solitons and metrics with half harmonic Weyl curvature are two natural generalizations of Einstein metrics on four-manifolds. In this paper we prove that if a metric has structures of both gradient shrinking Ricci soliton and half harmonic Weyl curvature, then except for three examples, it has to be an Einstein metric with positive scalar curvature. Precisely, we prove that a four-dimensional gradient shrinking Ricci soliton with $\delta W^{\pm} = 0$ is either Einstein, or a finite quotient of $S^3 \times \mathbb{R}$, $S^2 \times \mathbb{R}^2$ or $\mathbb{R}^4$. We also prove that a four-dimensional gradient Ricci soliton with constant scalar curvature is either Kähler–Einstein, or a finite quotient of $M \times \mathbb{C}$, where M is a Riemann surface. The method of our proof is to construct a weighted subharmonic function using curvature decompositions and the Weitzenböck formula for half Weyl curvature, and the method was motivated by previous work (Gursky and LeBrun in Ann Glob Anal Geom 17:315–328, 1999; Wu in Einstein four-manifolds of three-nonnegative curvature operator 2013; Trans Am Math Soc 369:1079–1096, 2017; Yang in Invent Math 142:435–450, 2000) on the rigidity of Einstein four-manifolds with positive sectional curvature, and previous work (Cao and Chen in Trans Am Math Soc 364:2377–2391, 2012; Duke Math J 162:1003–1204, 2013; Catino in Math Ann 35:629–635, 2013) on the rigidity of gradient Ricci solitons.

Mathematics Subject Classification Primary 53C24 · 53C25

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Mathematische Annalen

Heat kernel on smooth metric measure spaces and applications

Jia-Yong Wu[1] · Peng Wu[2]

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Abstract We derive a Harnack inequality for positive solutions of the f-heat equation and Gaussian upper and lower bound estimates for the f-heat kernel on complete smooth metric measure spaces with Bakry–Émery Ricci curvature bounded below. Both upper and lower bound estimates are sharp when the Bakry–Émery Ricci curvature is nonnegative. The main argument is the De Giorgi–Nash–Moser theory. As applications, we prove an L^1_f-Liouville theorem for f-subharmonic functions and an L^1_f-uniqueness theorem for f-heat equations when f has at most linear growth. We also obtain eigenvalues estimates and f-Green's function estimates for the f-Laplace operator.

Mathematics Subject Classification Primary 35K08; Secondary 53C21 · 58J35

1 Introduction

This is a sequel to our earlier work [49], we investigate heat kernel estimates on smooth metric measure spaces. For Riemannian manifolds, there are two classical methods for heat kernel estimates. One is the gradient estimate technique developed by Li and Yau [27], which they used to derive two-sided Gaussian bounds for the heat kernel on Riemannian manifolds with Ricci curvature bounded below. The other is the

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Mathematische Zeitschrift

Elliptic gradient estimates for a weighted heat equation and applications

Jia-Yong Wu

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Abstract We obtain two elliptic gradient estimates for positive solutions to the f-heat equation on a complete smooth metric measure space with only Bakry–Émery Ricci tensor bounded below. One is a local sharp Souplet–Zhang's type and the other is a global Hamilton's type. As applications, we prove parabolic Liouville theorems for ancient solutions satisfying some growth restriction near infinity. In particular the Liouville results are suitable for the gradient shrinking or steady Ricci solitons. The estimates of derivation of the f-heat kernel are also obtained.

Keywords Gradient estimate · Liouville theorem · Smooth metric measure space · Bakry–Émery Ricci tensor · Ricci soliton · Heat equation · Heat kernel

Mathematics Subject Classification Primary 58J35; Secondary 35B53 · 35K05

1 Introduction

Let (M, g) be an n-dimensional complete Riemannian manifold, and let f be a smooth function on M. Then the triple $(M, g, e^{-f}dv)$ is called a complete smooth metric measure space, where dv is the volume element of g, and $e^{-f}dv$ is called the weighted measure. On $(M, g, e^{-f}dv)$, the m-Bakry–Émery Ricci tensor [3] is defined by

$$Ric_f^m := Ric + \nabla^2 f - \frac{1}{m} df \otimes df$$

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Mathematische Annalen

Heat kernel on smooth metric measure spaces with nonnegative curvature

Jia-Yong Wu · Peng Wu

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Abstract We derive a local Gaussian upper bound for the f-heat kernel on complete smooth metric measure space $(M, g, e^{-f}dv)$ with nonnegative Bakry–Émery Ricci curvature. As applications, we obtain a sharp L^1_f-Liouville theorem for f-subharmonic functions and an L^1_f-uniqueness property for nonnegative solutions of the f-heat equation, assuming f is of at most quadratic growth. In particular, any L^1_f-integrable f-subharmonic function on gradient shrinking and steady Ricci solitons must be constant. We also provide explicit f-heat kernel for Gaussian solitons.

Mathematics Subject Classification Primary 35K08; Secondary 53C21 · 58J35

1 Introduction and main results

In this paper we study Gaussian upper estimates for the f-heat kernel on smooth metric measure spaces with nonnegative Bakry–Émery Ricci curvature and their applications. Recall that a complete smooth metric measure space is a triple $(M, g, e^{-f}dv)$, where (M, g) is an n-dimensional complete Riemannian manifold, dv is the volume element of g, f is a smooth function on M, and $e^{-f}dv$ (for short, $d\mu$) is called the weighted volume element or the weighted measure. The m-Bakry–Émery Ricci curvature [1] associated to $(M, g, e^{-f}dv)$ is defined by

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L^p-Liouville theorems on complete smooth metric measure spaces ☆

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Abstract

We study some function-theoretic properties on a complete smooth metric measure space $(M, g, e^{-f} dv)$ with Bakry–Émery Ricci curvature bounded from below. We derive a Moser's parabolic Harnack inequality for the f-heat equation, which leads to upper and lower Gaussian bounds on the f-heat kernel. We also prove L^p-Liouville theorems in terms of the lower bound of Bakry–Émery Ricci curvature and the bound of function f, which generalize the classical Ricci curvature case and the N-Bakry–Émery Ricci curvature case.

MSC: primary 53C21; secondary 58J35

Keywords: Bakry–Émery Ricci curvature; f-Laplacian; f-heat kernel; Harnack inequality; Liouville theorem

1. Introduction and main results

1.1. Background

Let (M, g) be an n-dimensional complete Riemannian manifold and f be a smooth function on M. We define a symmetric diffusion operator Δ_f (or f-Laplacian), which is given by

$$\Delta_f := \Delta - \nabla f \cdot \nabla,$$

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DIFFERENTIAL HARNACK INEQUALITIES FOR NONLINEAR HEAT EQUATIONS WITH POTENTIALS UNDER THE RICCI FLOW

JIA-YONG WU

We prove several differential Harnack inequalities for positive solutions to nonlinear backward heat equations with different potentials coupled with the Ricci flow. We also derive an interpolated Harnack inequality for the nonlinear heat equation under the ε-Ricci flow on a closed surface. These new Harnack inequalities extend the previous differential Harnack inequalities for linear heat equations with potentials under the Ricci flow.

1. Introduction and main results

Background. The study of differential Harnack estimates for parabolic equations originated with the work of P. Li and S.-T. Yau [1986], who first proved a gradient estimate for the heat equation via the maximum principle (though a precursory form of their estimate appeared in [Aronson and Bénilan 1979]). Using their gradient estimate, the same authors derived a classical Harnack inequality by integrating the gradient estimate along space-time paths. This result was generalized to Harnack inequalities for some nonlinear heat-type equations in [Yau 1994] and for some non-self-adjoint evolution equations in [Yau 1995]. Recently, J. Li and X. Xu [2011] gave sharper local estimates than previous results for the heat equation on Riemannian manifolds with Ricci curvature bounded below. Surprisingly, R. Hamilton employed similar techniques to obtain Harnack inequalities for the Ricci flow [Hamilton 1993a], and the mean curvature flow [Hamilton 1995]. In dimension two, a differential Harnack estimate for the positive scalar curvature was proved in [Hamilton 1988], and then extended by B. Chow [1991a] when the scalar curvature changes sign. Similar techniques were used to obtain the Harnack inequalities for the Gauss curvature flow [Chow 1991b] and the Yamabe flow [Chow 1992]. H.-D. Cao [1992] proved a Harnack inequality for the Kähler–Ricci

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ORIGINAL PAPER

First eigenvalue of the p-Laplace operator along the Ricci flow

Jia-Yong Wu · Er-Min Wang · Yu Zheng

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Abstract In this article, we mainly investigate continuity, monotonicity and differentiability for the first eigenvalue of the p-Laplace operator along the Ricci flow on closed manifolds. We show that the first p-eigenvalue is strictly increasing and differentiable almost everywhere along the Ricci flow under some curvature assumptions. In particular, for an orientable closed surface, we construct various monotonic quantities and prove that the first p-eigenvalue is differentiable almost everywhere along the Ricci flow without any curvature assumption, and therefore derive a p-eigenvalue comparison-type theorem when its Euler characteristic is negative.

Keywords Ricci flow · First eigenvalue · p-Laplace operator · Continuity · Monotonicity · Differentiability

Mathematics Subject Classification (2000) Primary 58C40 · Secondary 53C44

1 Introduction

Given a compact Riemannian manifold (M^n, g_0) without boundary, the Ricci flow is the following evolution equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{1.1}$$

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