Smooth crossed product of minimal unique ergodic
diffeomorphism of odd sphere

Hongzhi Liu*

Abstract. For minimal unique ergodic diffeomorphisms $\alpha_n$ of $S^{2n+1}(n > 0)$ and $\alpha_m$ of $S^{2m+1}(m > 0)$, the $C^*$-crossed product algebra $C(S^{2n+1}) \rtimes_{\alpha_n} \mathbb{Z}$ is isomorphic to $C(S^{2m+1}) \rtimes_{\alpha_m} \mathbb{Z}$ even though $n \neq m$. However, by cyclic cohomology, we show that smooth crossed product algebra $C^\infty(S^{2n+1}) \rtimes_{\alpha_n} \mathbb{Z}$ is not isomorphic to $C^\infty(S^{2m+1}) \rtimes_{\alpha_m} \mathbb{Z}$ if $n \neq m$.


Keywords. Smooth crossed products, cyclic cohomology.

1. Introduction

$C^*$-algebra classification theory can be used to study dynamical systems. Pimsner, Voiculescu [16] and Rieffel [17] proved that two irrational rotation dynamical systems are flip conjugate to each other if and only if their corresponding irrational rotation $C^*$-algebras are isomorphic to each other. Giordano, Putnam and Skau have shown that the minimal dynamical systems of Cantor set can be completely classified by $C^*$-crossed product algebras up to strong orbit equivalence [6]. See [8–11,19] for more examples.

However, there are examples of different minimal diffeomorphisms give the same $C^*$-algebras. Let $\alpha_l$ be minimal unique ergodic diffeomorphism of $S^{2l+1}$, $l = 1, 2, \ldots$. It is well known that the ordered $K$-theory of $C(S^{2n+1}) \rtimes_{\alpha_n} \mathbb{Z}$ and $C(S^{2m+1}) \rtimes_{\alpha_m} \mathbb{Z}$ are isomorphic to each other [13]. This implies

$$C(S^{2n+1}) \rtimes_{\alpha_n} \mathbb{Z} \cong C(S^{2m+1}) \rtimes_{\alpha_m} \mathbb{Z}$$

no matter if $n = m$ or not according to a theory of Toms and Winter [19] and Phillips [13]. See [15] and [14] for more examples.

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Mutual embeddability equivalence relation for rotation algebras

Bingzhe Hou\textsuperscript{a,\,*}, Hongzhi Liu\textsuperscript{b}, Xiaotian Pan\textsuperscript{a}
\textsuperscript{a} School of Mathematics, Jilin University, 130012, Changchun, PR China
\textsuperscript{b} Shanghai Center for Mathematical Sciences, 200433, Shanghai, PR China

ABSTRACT

Mutual embeddability is an equivalence relation in $C^*$-algebras. In this paper, we characterize the classification of rotation algebras in the sense of mutual embeddability.

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1. Introduction

Let $\theta$ be a real number in $[0, 1)$. Recall that the rotation algebra $A_\theta$ is the universal $C^*$-algebra generated by two unitaries $u$ and $v$ satisfying the Heisenberg relation

$$uv = e^{2\pi i \theta} vu.$$

If $\theta = \frac{q}{p}$ with $(p, q) = 1$, $A_\theta$ is a $C^*$-algebra of type I with spectrum $T^2$, and the dimension of any irreducible representation of $A_\theta$ is $p$. All tracial states of rational rotation algebra induce the same map from $K_0(A_\theta)$ to $\mathbb{R}$ as shown in [7]. If $\theta$ is irrational, $A_\theta$ is simple, and has unique tracial state. Moreover, it was shown in [8] that irrational rotation algebras are AT algebras.

Rotation algebra $A_\theta$ is $\dagger$-isomorphic to the crossed product $C^*$-algebra $C(T) \rtimes_{f_\theta} \mathbb{Z}$, where $f_\theta$ is the rotation map on the unit circle $T$ defined by

$$f_\theta(z) = e^{2\pi i \theta} z, \quad \text{for any } z \in T.$$

For convenience, we will not differ the notations $A_\theta$ and $C(T) \rtimes_{f_\theta} \mathbb{Z}$.  

\* Corresponding author.
E-mail addresses: houbz@jlu.edu.cn (B. Hou), lhz3012@gmail.com (H. Liu), panxt15@mails.jlu.edu.cn (X. Pan).

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TWO MINIMAL UNIQUE ERGODIC Diffeomorphisms on a Manifolds AND THEIR SMOOTH Crossed Product Algebras

HONGZHI LIU
EMAIL: 1063733099@QQ.COM

Abstract. In this article we construct two minimal unique ergodic diffeomorphisms \(\alpha\) and \(\beta\) on \(S^1 \times S^6 \times S^8\). We will show that \(C(S^1 \times S^6 \times S^8) \rtimes_\alpha \mathbb{Z}\) and \(C(S^1 \times S^6 \times S^8) \rtimes_\beta \mathbb{Z}\) are equivalent to each other, while \(C^\infty(S^1 \times S^6 \times S^8) \rtimes_\alpha \mathbb{Z}\) and \(C^\infty(S^1 \times S^6 \times S^8) \rtimes_\beta \mathbb{Z}\) are not.

Keywords: smooth crossed products, cyclic cohomology.
MSC: 46L80, 46L87.

1. Introduction

Isomorphism between two irrational rotation algebras implies flip equivalence between their corresponding irrational rotation transformations ([15], [16]). Giordano, Putnam and Skau have given a classification of dynamical systems on Cantor set based on \(C^*\)-crossed product algebras ([5]).

However, different dynamical systems may give equivalent \(C^*\)-crossed product algebras (see [13], [14] for examples). It is interesting to investigate smooth crossed product algebra in these cases. Let \(g\) and \(h\) be minimal unique ergodic diffeomorphisms of \(S^{2m+1}\) and \(S^{2n+1}\) respectively, with \(m, n \geq 1\). In [12], N. C. Phillips proved that \(C(S^{2m+1}) \rtimes_g \mathbb{Z} \cong C(S^{2n+1}) \rtimes_h \mathbb{Z}\). In [2], the author proved that 
\[C^\infty(S^{2m+1}) \rtimes_g \mathbb{Z} \not\cong C^\infty(S^{2n+1}) \rtimes_h \mathbb{Z}\] 
by checking the grading structure of cyclic cohomology. People may argue that this is too obvious since these two diffeomorphisms are of different manifolds. In this article we construct two diffeomorphisms of a same manifold giving same \(C^*\)-crossed product algebra and different smooth crossed product algebras.

We introduce several notions and theories important for our construction in the next section. In the third section we give the construction (Theorem 3.1) of minimal unique ergodic diffeomorphisms \(\alpha\) and \(\beta\) of \(C(S^1 \times S^6 \times S^8)\). The fourth and fifth section are devoted to computation of \(K\)-theory and cyclic cohomology. These computation together show that
\[C(S^1 \times S^6 \times S^8) \rtimes_\alpha \mathbb{Z} \cong C(S^1 \times S^6 \times S^8) \rtimes_\beta \mathbb{Z},\]
\[C^\infty(S^1 \times S^6 \times S^8) \rtimes_\alpha \mathbb{Z} \not\cong C^\infty(S^1 \times S^6 \times S^8) \rtimes_\beta \mathbb{Z}.\]

In the last section we prove Theorem 5.1.

2. Preliminary

Let \(M\) be a finite dimensional compact manifold. Choose finitely many vector fields \(X_1, X_2, \ldots, X_n\) on \(M\) which can span the tangent space at any point (\(n\) is not necessarily equal to the dimension of \(M\)). Define seminorms \(\|\cdot\|_n\):
\[\|f\|_n = \sum_{1 \leq k_1 \leq \cdots \leq k_n \leq n} \|X_{k_1}X_{k_2} \cdots X_{k_n}f\|_\infty, n \in \mathbb{Z}_+ \cup \{0\}, f \in C^\infty(M).\]

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