Upper bounds for geodesic periods over hyperbolic manifolds

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Received 23 June 2016
Accepted 25 December 2017
Published 25 January 2018

We prove an upper bound for geodesic periods of Maass forms over hyperbolic manifolds. By definition, such periods are integrals of Maass forms restricted to a special geodesic cycle of the ambient manifold, against a Maass form on the cycle. Under certain restrictions, the bound will be uniform.

Keywords: Geodesic period; Maass form; hyperbolic manifold.

Mathematics Subject Classification 2010: 22E43, 11F03, 11F70

1. Introduction

Let \( X \) be a \( d \)-dimensional connected complete hyperbolic manifold with finite volume, \( \phi \) a square integrable Laplace eigenfunction on \( X \), whose eigenvalue is denoted by \( \lambda \geq 0 \). In the theory of automorphic forms, \( \phi \) is also called “Maass form” (after \[12\]). We normalize \( \phi \) so that it has \( L^2 \)-norm 1. Let \( Y \) be a special cycle of \( X \) which is compact, totally geodesic and has codimension 1 (see Sec. 2 for the precise description). Fix a hyperbolic measure \( \text{d}y \) on \( Y \). Given a normalized Maass form \( \psi \) on \( Y \) with the Laplace eigenvalue \( \mu \geq 0 \), define the period integral

\[
P_Y(\phi, \psi) := \int_Y \phi(y)\psi(y)\text{d}y.
\]

This integral converges since \( Y \) is compact and \( \phi, \psi \) are smooth (by the elliptic regularity theorem). We call \( P_Y(\phi, \psi) \) geodesic period from the geometric perspective. Such a period fits into the general notion of automorphic period which plays a central role in the study of automorphic forms thanks to its close relations with automorphic representations and special values of certain automorphic \( L \)-functions (see \[23\] and references therein).

The aim of this paper is to prove an upper bound for geodesic periods.
LATTICE POINTS COUNTING AND BOUNDS ON PERIODS OF MAASS FORMS

ANDRE REZNIKOV AND FENG SU

Abstract. We provide a “soft” proof for non-trivial bounds on spherical, hyperbolic and unipotent Fourier coefficients of a fixed Maass form for a general co-finite lattice \( \Gamma \) in \( \text{PGL}_2(\mathbb{R}) \). We use the amplification method based on the Airy type phenomenon for corresponding matrix coefficients and an effective Selberg type pointwise asymptotic for the lattice points counting in various homogeneous spaces for the group \( \text{PGL}_2(\mathbb{R}) \). This requires only \( L^2 \)-theory. We also show how to use the uniform bound for the \( L^4 \)-norm of \( K \)-types in a fixed automorphic representation of \( \text{PGL}_2(\mathbb{R}) \) obtained in [BR2] in order to slightly improve these bounds.

1. Introduction

In this paper, we explore an interplay between two classical topics in analytic theory of automorphic functions. Namely, we show how to use lattice points counting in various \( \text{PGL}_2(\mathbb{R}) \)-homogeneous spaces in order to obtain non-trivial bounds on Fourier coefficients of Maass forms (or more generally, bounds on generalized periods of Maass forms). We start with an analysis of certain (generalized) matrix coefficients of unitary irreducible representations of the group \( G = \text{PGL}_2(\mathbb{R}) \), that is, functions \( m_{v,l}(g) = \langle \pi(g)v, l \rangle \) on the group \( G \), where \( v \in V \) is a vector in an irreducible unitary representation \( (\pi, V) \) of \( G \) and \( l \in V^* \) is a (generalized) vector in the dual representation. Such matrix coefficients have well-understood and uniform asymptotic behavior for large elements \( g \in G \) (e.g., uniform decay for matrix coefficients of tempered representations at infinity). Our point of departure is that, in spite of this fact, in certain ranges of parameters (i.e., parameters of the representation and the group variable) matrix coefficients might exhibit “abnormally” large values. The origin of this phenomenon is classical and best associated with the Airy function (see [A], [He], [M]). We use this property of matrix coefficients in order to “amplify” (in the language of analytic number theory) certain automorphic coefficients. A somewhat unexpected fact is that the Airy type phenomenon for matrix coefficients holds (for appropriate values of parameters) over domains in \( \text{PGL}_2(\mathbb{R}) \) with a large volume. This allows us to use as a global input an effective...
Research Article

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Upper bounds for geodesic periods over rank one locally symmetric spaces

https://doi.org/10.1515/forum-2017-0185
Received September 4, 2017; revised November 29, 2017

Abstract: We prove upper bounds for geodesic periods of automorphic forms over general rank one locally symmetric spaces. Such periods are integrals of automorphic forms restricted to special totally geodesic cycles of the ambient manifold and twisted with automorphic forms on the cycles. The upper bounds are in terms of the Laplace eigenvalues of the two automorphic forms, and they generalize previous results for real hyperbolic manifolds to the context of all rank one locally symmetric spaces.

Keywords: Locally symmetric spaces, automorphic forms, geodesic periods, totally geodesic cycles, unitary representations, reductive Lie groups

MSC 2010: 11F70, 22E46, 53C35

Communicated by: Jan Bruinier

1 Introduction

Estimating the restriction of a Laplace eigenfunction $f$ on a Riemannian manifold $X$ to a compact submanifold $Y$ is a classical problem in partial differential equations and global analysis. There are various types of restriction problems, for example, one may estimate $L^p$-norms ($0 < p \leq \infty$) of the restriction $f|_Y$ in terms of the eigenvalue of $f$. A more refined quantity in the case $p = 2$ are the Fourier coefficients of $f|_Y$ with respect to an orthonormal basis of Laplace eigenfunctions $g$ in $L^2(Y)$. Each Fourier coefficient is given by an integral $P_Y(f,g)$ of $f|_Y$ against $g$, and one can ask for estimates in terms of the eigenvalues of $f$ and $g$.

In the context of locally symmetric spaces, the integrals $P_Y(f,g)$ are also called periods and they carry important arithmetic information of $f$ and $g$. One instance of this is when $X$ is an arithmetic hyperbolic surface and $Y$ a closed geodesic, in which case the period integrals $P_Y(f,g)$ are proportional to special $L$-values via Waldspurger’s formula. In a series of papers, J. Bernstein and A. Reznikov [2–4, 17, 18] studied several types of period integrals for automorphic forms on hyperbolic surfaces in connection with the representation theory of the corresponding isometry groups of the universal coverings of $X$ and $Y$. More recently, their techniques were generalized to the case of higher-dimensional real hyperbolic manifolds by the authors [14, 19], where estimates for $P_Y(f,g)$ either in terms of the eigenvalue of $f$ or in terms of the eigenvalue of $g$ were obtained.

In this work, we extend the techniques by Bernstein and Reznikov even further and study period integrals for arbitrary locally symmetric spaces $X$ of rank one. In this context, we obtain estimates for $P_Y(f,g)$ in terms of the eigenvalues of $f$ and $g$, for compact totally geodesic cycles $Y$ of a particular form. This includes real, complex and quaternionic hyperbolic manifolds $X$ as well as quotients of the 16-dimensional octonionic plane in which case $Y$ is an 8-dimensional real hyperbolic submanifold.

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