Wigner-type theorem on transition probability preserving maps in semifinite factors

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\textbf{A B S T R A C T}

The Wigner’s theorem, which is one of the cornerstones of the mathematical formulation of quantum mechanics, asserts that every symmetry of quantum system is unitary or anti-unitary. This classical result was first given by Wigner in 1931. Thereafter it has been proved and generalized in various ways by many authors. Recently, G.P. Gehér extended Wigner’s and Molnár’s theorems and characterized the transformations on the Grassmann space of all rank-$n$ projections which preserve the transition probability. The aim of this paper is to provide a new approach to describe the general form of the transition probability preserving (not necessarily bijective) maps between Grassmann spaces. As a byproduct, we are able to generalize the results of Molnár and G.P. Gehér.

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ABSTRACT

In the paper, we prove an analogue of the Kato–Rosenblum theorem in a semifinite von Neumann algebra. Let $\mathcal{M}$ be a countably decomposable, properly infinite, semifinite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $\tau$ be a faithful normal semifinite tracial weight of $\mathcal{M}$. Suppose that $H$ and $H_1$ are self-adjoint operators affiliated with $\mathcal{M}$. We show that if $H - H_1$ is in $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$, then the norm absolutely continuous parts of $H$ and $H_1$ are unitarily equivalent. This implies that the real part of a non-normal hyponormal operator in $\mathcal{M}$ is not a perturbation by $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$ of a diagonal operator.

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A new class of Kadison–Singer algebras

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\textbf{Abstract}

We show that the projection lattice generated by a maximal nest and a rank one projection in a separable infinite-dimensional Hilbert space is in general reflexive. Moreover we show that the corresponding reflexive algebra has a maximal triangular property, equivalently, it is a Kadison–Singer algebra. Similar results are also obtained for the lattice generated by a finite nest and a projection in a finite factor.

\textbf{1. Introduction}

The development of the theory of non-self-adjoint operator algebras is parallel to that of the self-adjoint theory. The maximal triangular algebras introduced by Kadison and Singer \cite{11} and the reflexive algebras are two important classes of non-self-adjoint operator algebras. Many important results obtained in non-self-adjoint algebras depend on relations to compact operators which are almost absent in the self-adjoint theory. Therefore there is no fruitful interaction between these two theories in the past. In order to use the powerful tools in self-adjoint operator algebras, Liming Ge and Wei Yuan \cite{5} introduced a new class of non-self-adjoint algebras, Kadison–Singer algebras, which are reflexive and maximal with respect to their diagonals. The corresponding reflexive lattice is called Kadison–Singer lattice. Kadison–Singer algebras combine triangularity, reflexivity and von Neumann algebra properties together and makes the techniques of self-adjoint operator algebras more involved in the study of non-self-adjoint operator algebras. The results in \cite{5} and \cite{6} also establish surprising connections between classical geometry and non-self-adjoint operator algebras.

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On the properties of some sets of von Neumann algebras under perturbation

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Abstract
Let $\mathcal{L}$ be a type II$_1$ factor with separable predual and $\tau$ be a normal faithful tracial state of $\mathcal{L}$. We first show that the set of subfactors of $\mathcal{L}$ with property $\Gamma$, the set of type II$_1$ subfactors of $\mathcal{L}$ with similarity property and the set of all McDuff subfactors of $\mathcal{L}$ are open and closed in the Hausdorff metric $d_2$ induced by the trace norm; then we show that the set of all hyperfinite von Neumann subalgebras of $\mathcal{L}$ is closed in $d_2$. We also consider the connection of perturbation of operator algebras under $d_2$ with the fundamental group and the generator problem of type II$_1$ factors. When $\mathcal{M}$ is a finite von Neumann algebra with a normal faithful trace, the set of all von Neumann subalgebras $\mathcal{B}$ of $\mathcal{M}$ such that $\mathcal{B} \subset \mathcal{M}$ is rigid is closed in the Hausdorff metric $d_2$.

Keywords
type II$_1$ factor, property $\Gamma$, McDuff factor, hyperfinite, similarity length

MSC(2010)
46L10, 46L50


1 Introduction and preliminaries

Kadison and Kastler [12] introduced the study of uniform perturbations of operator algebras. They considered a fixed $C^*$-algebra $\mathcal{C}$ and equipped the set of all $C^*$-subalgebras of $\mathcal{C}$ with a metric arising from Hausdorff distance between the unit balls of these subalgebras. We first recall the following definition of the metric $d$ defined on the set of all $C^*$-subalgebras of a $C^*$-algebra $\mathcal{C}$ (see [12]).

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-subalgebras of a $C^*$-algebra $\mathcal{C}$. Define $d(\mathcal{A}, \mathcal{B})$ to be the infimum of all $\gamma > 0$ with the property that given $x$ in the unit ball of $\mathcal{A}$ or $\mathcal{B}$, there exists $y$ in the unit ball of the other algebra with $\|x - y\| < \gamma$.

It was conjectured by Kadison and Kastler [12] that sufficiently close von Neumann algebras (or $C^*$-algebras) are necessarily unitarily conjugate. The first positive answer to Kadison-Kastler’s conjecture was given by Christensen [11] and Phillips [15] when either $\mathcal{A}$ or $\mathcal{B}$ is a von Neumann algebra of type I. Many results related to this conjecture have been obtained during the past 40 years (see [1–4, 8, 9]). One-sided versions of Kadison-Kastler’s conjecture was introduced and studied by Christensen as well. Christensen [6] proved that a nuclear $C^*$-algebra that is nearly contained in an injective von Neumann algebra is unitarily conjugate to this von Neumann algebra. Christensen et al. [7] showed that the property of having a positive answer to Kadison’s similarity problem transfers to close $C^*$-algebras. Very recently, Kadison-Kastler’s conjecture has been proved for the class of separable nuclear $C^*$-algebras in the remarkable paper [9].

Christensen [5] also initiated the study of perturbation of von Neumann subalgebras in a finite von Neumann algebra in the Hausdorff metric with respect to the trace norm.
A Note on the Perturbations of Compact Quantum Metric Spaces

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Abstract In this short note, we consider the perturbation of compact quantum metric spaces. We first show that for two compact quantum metric spaces \((\mathcal{A}, P)\) and \((\mathcal{B}, Q)\) for which \(\mathcal{A}\) and \(\mathcal{B}\) are subspaces of an order-unit space \(\mathcal{C}\) and \(P\) and \(Q\) are Lip-norms on \(\mathcal{A}\) and \(\mathcal{B}\) respectively, the quantum Gromov–Hausdorff distance between \((\mathcal{A}, P)\) and \((\mathcal{B}, Q)\) is small under certain conditions. Then some other perturbation results on compact quantum metric spaces derived from spectral triples are also given.

Keywords Compact quantum metric space, quantum Gromov–Hausdorff distance, C*-algebra, Lip-norm, spectral triple

MR(2010) Subject Classification 46L30, 46L89

1 Introduction and Preliminaries

Noncommutative metric geometry is the study of noncommutative generalizations of compact metric spaces. Inspired by the work of Connes [1, 2], Rieffel introduced the notion of a compact quantum metric space in [11, 12] and in [13] he introduced a generalization of the Gromov–Hausdorff distance between compact metric spaces (see [6]). Rieffel provided in [14] a meaning to many approximations of classical and quantum spaces by matrix algebras found in the physics literature (see for instance [3]) and developed a new set of techniques in the study of the geometry of C*-algebras (see [15] for example). For background of C*-algebras we refer to [7]. We remark that there are lots of results related to Rieffel’s distance ([4, 8–10, 16]).

We first recall some definitions and results from Rieffel’s memoir [13].

**Definition 1.1** An order-unit space \((\mathcal{A}, I)\) is a real partially ordered vector space \(\mathcal{A}\), together with a distinguished element \(I\) (the order unit) which satisfies

1. **(Order unit property)** For each \(a \in \mathcal{A}\), there is an \(r \in \mathbb{R}\) such that \(a \leq rI\).
2. **(Archimedean property)** If \(a \in \mathcal{A}\) and if \(a \leq rI\) for all \(r \in \mathbb{R}\) with \(r \geq 0\), then \(a \leq 0\).

Note that for each \(a\) in the order-unit space \((\mathcal{A}, I)\),

\[\|a\| = \inf \{r \in \mathbb{R} : -rI \leq a \leq rI\}\]

defines a norm on \(\mathcal{A}\). For an element \(a\) in \(\mathcal{A}\), we also let

\[\|\tilde{a}\| = \inf_{\lambda \in \mathbb{R}} \|a - \lambda I\|\]

denote the quotient norm in \(\mathcal{A}/\mathbb{R}I\).
关于 Haar 邑元的一个注记

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摘　要　设 H 是一个 II_1 型因子，τ 是 H 的正规的、忠实的迹态，U ∈ H 是一个 Haar 邑元，p ∈ H 是一个投影，w 是一个 Haar 邑元且 w 和 h 是自由的。我们还给出了 pUp 的极分解。

关键词　Haar 邑元　II_1 型因子　迹态　自由　极分解

MSC (2000) 主题分类　46L10, 46L50

1 引言

von Neumann 代数由 von Neumann 在 1929 年引入。设 H 是一个复 Hilbert 空间。我们用 B(H) 表示 H 上所有有界线性算子构成的集合。若 M ⊆ B(H)，M 为 *- 代数（即若 A ∈ M，则 A* ∈ M），则称 M 是一个 von Neumann 代数。我们首先给出自由概率论中的基本概念（见 [4, 7, 8]）。

定义 1.1 设 (H, τ) 是一个 W* 概率空间，I 是一个固定的指标集，{A_i : i ∈ I} 是 H 的一族有单位的子代数。若对每个 n ∈ N，1 ≤ j ≤ n，i_j ∈ I，i_1 ≠ i_2 ≠ ··· ≠ i_n，A_i_j ∈ H, i_j，τ(A_i_j) = 0，则称 {A_i : i ∈ I} 关于 τ 是自由的。设 X_1, X_2, ..., X_n 为 H 中的随机变量，A_i 是由 X_i 生成的子代数。若 A_1, ..., A_n 是自由的，则称 X_1, ..., X_n 关于 τ 是自由的。

定义 1.2 设 (H, τ) 是一个 W* 概率空间，U 是 H 中的一个 Haar 邑元。若对所有的 n ∈ Z, n ≠ 0, 都有 τ(U(n)) = 0，则称 U 为一个 Haar 邑元。

Haar 邑元在自由概率论中起着重要的作用（见 [4]）。下面我们给出 Haar 邑元的一个例子。

例 1.3 假设 F_2 是由两个生成元 g_1 和 g_2 (非交换) 自由群。设 λ 是 F_2 在 Hilbert 空间 L^2(F_2) 上的左正则表示。设 L^2(F_2) 是由 {λ(g) : g ∈ F_2} 生成的 von Neumann 代数。设 x_e 是 F_2 上
WEAK-2-LOCAL ISOMETRIES ON UNIFORM ALGEBRAS AND LIPSCHITZ ALGEBRAS

LEI LI, ANTONIO M. PERALTA, LIGUANG WANG, AND YA-SHU WANG

Abstract: We establish spherical variants of the Gleason–Kahane–Żelazko and Kowalski–Słodkowski theorems, and we apply them to prove that every weak-2-local isometry between two uniform algebras is a linear map. Among the consequences, we solve a couple of problems posed by O. Hatori, T. Miura, H. Oka, and H. Takagi in 2007.

Another application is given in the setting of weak-2-local isometries between Lipschitz algebras by showing that given two metric spaces $E$ and $F$ such that the set $\text{Iso}(\text{Lip}(E), \| \cdot \|, (\text{Lip}(F), \| \cdot \|))$ is canonical, then every weak-2-local Iso$(\text{Lip}(E), \| \cdot \|, (\text{Lip}(F), \| \cdot \|))$-map $\Delta$ from $\text{Lip}(E)$ to $\text{Lip}(F)$ is a linear map, where $\| \cdot \|$ can indistinctly stand for $\|f\|_L := \max\{L(f), \|f\|_{\infty}\}$ or $\|f\|_* := L(f) + \|f\|_{\infty}$.

2010 Mathematics Subject Classification: Primary: 46B04, 46B20, 46J10, 46E15; Secondary: 30H05, 32A38, 46J15, 47B48, 47B38, 47D03.

Key words: 2-local isometries, uniform algebras, Lipschitz functions, spherical Gleason–Kahane–Żelazko theorem, spherical Kowalski–Słodkowski theorem, weak-2-local isometries.

1. Introduction

Let $\text{Is}(X, Y)$ denote the set of all surjective linear isometries between two Banach spaces $X$ and $Y$. Clearly $\text{Is}(X, Y)$ can be regarded as a subset of the space $L(X, Y)$ of all linear maps between $X$ and $Y$. We shall write $\text{Is}(X)$ instead of $\text{Is}(X, X)$. Accordingly to the notation in [12, 13, 39, 38] and [42], we shall say that a (not-necessarily linear nor continuous) mapping $\Delta: X \to Y$ is a weak-2-local $\text{Is}(X, Y)$-map or a weak-2-local isometry (respectively, a 2-local $\text{Is}(X, Y)$-map or a 2-local isometry) if for each $x, y \in X$ and $\phi \in Y^*$, there exists $T_{x, y, \phi}$ in $\text{Is}(X, Y)$, depending on $x$, $y$, and $\phi$ (respectively, for each $x, y \in X$, there exists $T_{x, y}$ in $\text{Is}(X, Y)$, depending on $x$ and $y$), satisfying

$$\phi \Delta(x) = \phi T_{x, y, \phi}(x) \text{ and } \phi \Delta(y) = \phi T_{x, y, \phi}(y)$$

(respectively, $\Delta(x) = T_{x, y}(x)$ and $\Delta(y) = T_{x, y}(y)$). A Banach space $X$ is said to be (weak-2-iso-reflexive if every (weak-)2-local isometry on $X$ is both linear and surjective.
Von Neumann algebras as complemented subspaces of $B(\mathcal{H})$

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Let $\mathcal{M}$ be a von Neumann algebra of type II$_1$ which is also a complemented subspace of $B(\mathcal{H})$. We establish an algebraic criterion, which ensures that $\mathcal{M}$ is an injective von Neumann algebra. As a corollary we show that if $\mathcal{M}$ is a complemented factor of type II$_1$ on a Hilbert space $\mathcal{H}$, then $\mathcal{M}$ is injective if its fundamental group is nontrivial.

Keywords: Type II$_1$ factor; fundamental group; hyperfinite type II$_1$ factor; injective von Neumann algebra; complemented subspace.

Mathematics Subject Classification 2010: 46L10, 46L50

1. Introduction

In the early works [13–15] by Murray and von Neumann, they realized that there is a certain sort of rings of operators which to a large extent behave like the algebras $M_n(C)$ consisting of all complex $n \times n$ matrices, except that the natural dimension function now has the image $[0, 1]$ instead of the set $\{0, 1, \ldots, n\}$. Today rings of operators are called von Neumann algebras and the ones with a continuous dimension function with values in $[0, 1]$ are called von Neumann algebras of type II$_1$. Factors are von Neumann algebras whose centers consist of scalar multiples of the identity. Finite-dimensional factors are (isomorphic to) full matrix algebras. Infinite-dimensional factors admitting a positive and bounded trace are called factors of type II$_1$. Murray and von Neumann also realized that there are at least two non-isomorphic factors of type II$_1$, namely the free group factor $\mathcal{L}(F_2)$ and the hyperfinite type II$_1$ factor $\mathcal{R}$, where $\mathcal{L}(F_2)$ is the von Neumann algebra obtained by taking the ultraweak closure of the left regular representation of the non-abelian...


Property T for Actions

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Abstract  We study property T for an action \(\alpha\) of a discrete group \(\Gamma\) on a unital \(C^*\)-algebra \(\mathcal{A}\). Our main results improve some well-known results about property T for groups. Moreover, we introduce Hilbert \(\mathcal{A}\)-module property T and show that the action \(\alpha\) has property T if and only if the reduced crossed product \(\mathcal{A} \rtimes_{\alpha,r} \Gamma\) has Hilbert \(\mathcal{A}\)-module property T.

Keywords  Property T, \(\alpha\)-positive definite, \(\alpha\)-negative definite

MR(2010) Subject Classification  46L05, 46L55, 46L08

1 Introduction

Approximation theory is particularly important in group theory and operator algebra theory. There are many different approximation properties such as the Haagerup property, property T, the weak Haagerup property and so on. These approximation properties have been widely studied (eg. [2, 4, 6, 7, 9, 12, 13, 17–23]). Classical positive definite functions and conditionally negative definite functions on groups play important roles in characterizing the Haagerup property [8] and property T [5] of groups.

In 1967, Kazhdan [15] defined property T for locally compact groups in terms of unitary representations and proved that a large class of lattices are finitely generated. In fact, property T are very useful in many different fields such as differential geometry, ergodic theory, potential theory, operator algebras, combinatorics, computer science and the theory of algorithms. In particular, we can use property T to produce new examples related to the Baum–Connes conjecture.

Let \(\Gamma\) be a countable discrete group. We say that the group \(\Gamma\) has property (T) if whenever \(h_n: \Gamma \to \mathbb{C}\) is a sequence of positive definite functions converging to the constant function 1 pointwisely on \(\Gamma\), then \(h_n \to 1\) uniformly on \(\Gamma\).

In 1987, Anantharaman-Delaroche [1] introduced \(\alpha\)-positive definite functions associated to a \(C^*\)-dynamical system \((\mathcal{A}, \Gamma, \alpha)\). She also used this notion to characterize the amenability of actions of discrete groups on von Neumann algebras and on commutative \(C^*\)-algebras. Moreover, Dong and Ruan [10] showed that any \(\alpha\)-positive definite function on \(\Gamma\) taking values in the
W*- 三元算子环的摄动

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关键词 W*- 三元算子环; II$_1$ 型因子; $\Gamma$ 性质; McDuff 因子
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Perturbation of W*-ternary Ring of Operators

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Abstract We show that when the cb-distance $d_{cb}(V,W)$ between two W*-ternary ring of operators V and W is small, the distance between their linking von Neumann algebras $R(V)$ and $R(W)$ is also small. We show that W*-ternary ring of operators close to injective W*-ternary ring of operators is injective again. We also consider similar properties for W*-ternary ring of operators with property $\Gamma$ or McDuff property.

Keywords W*-ternary ring; factor of type II$_1$; property $\Gamma$; McDuff factor
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